On Frankl and Füredi's conjecture for 3-uniform hypergraphs

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Abstract

In 1965, Motzkin and Straus [5] provided a new proof of Turán's theorem based on a continuous characterization of the clique number of a graph using the Lagrangian of a graph. This new proof aroused interests in the study of Lagrangians of hypergraphs. In most applications, we need an upper bound for the Lagrangian of a hypergraph. Frankl and Füredi in [1] conjectured that the r-graph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest Lagrangian of all r-graphs with m edges. In this paper, we first establish some bounds for Lagrangians of some special 3-graphs, and then using these results, we prove that Frankl and Füredi's conjecture holds for 3-graphs with no more than 71 edges. Combining with previous results, we also confirm that Frankl and Füredi's conjecture holds for 3-graphs with m edges where $\binom{t-1}{3}-5 \leq m \leq \binom{t-1}{3}+\binom{t-2}{2}-(t-3)$.

Key Words: Cliques of Hypergraphs; Colex ordering; Lagrangians of r-uniform graphs; Optimization.

1 Introduction

In 1941, Turán [16] provided an answer to the following question: What is the maximum number of edges in a graph on n vertices without containing a complete graph of order k, for a given k? This is the well-known Turán theorem. Later, in another classical paper, Motzkin and Straus [5] provided a new proof of Turán's theorem based on a continuous characterization of the clique number of a graph using Lagrangians of graphs. This new proof aroused interests in the study of Lagrangians of r-graphs. The Lagrangian of a hypergraph has been a useful tool in hypergraph extremal problems. For example, Sidorenko [11] and Frankl-Furedi [1] applied Lagrangians of hypergraphs in finding Turán densities of hypergraphs. Frankl and Rödl [2] applied it in disproving Erdös' long standing jumping constant conjecture. More applications of Lagrangians can be found in [3], [6], and [12]. In most applications, we need an upper bound for the Lagrangian of a hypergraph. In the course of estimating Turán densities of some hypergraphs, Frankl and Füredi [1] asked the following question: Given $r \geq 3$ and $m \in \mathbb{N}$, how large can the Lagrangian of an r-graph with m edges be? Before stating their conjecture on this problem, we give some definitions and notations.

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For a set V and a positive integer r we denote by $V^{(r)}$ the family of all r-subsets of V. An r-uniform graph or r-graph G consists of a set $V(G)$ of vertices and a set $E(G) \subseteq V(G)^{(r)}$ of edges. An edge $e = \{a_1, a_2, \ldots, a_r\}$ will be simply denoted by $a_1a_2\ldots a_r$. An r-graph H is a *subgraph* of an r-graph G, denoted by $H \subseteq G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $\mathbb{N}$ be the set of all positive integers. For an integer $n \in \mathbb{N}$, we denote the set $\{1, 2, 3, \ldots, n\}$ by $[n]$. Let $K_t^{(r)}$ denote the complete r-graph on t vertices, that is the r-graph on t vertices containing all possible edges. A complete r-graph on t vertices is also called a clique with order t. We also let $[n]^{(r)}$ represent the complete r-graph on the vertex set $[n]$. When $r = 2$, an r-graph is a simple graph. When $r \geq 3$, an r-graph is often called a hypergraph.

Definition 1.1 *For an r-graph G with the vertex set $[n]$, edge set $E(G)$ and a vector $\vec{x} = (x_1, \ldots, x_n) \in R^n$, define*

$$\lambda(G, \vec{x}) = \sum_{i_1 i_2 \cdots i_r \in E(G)} x_{i_1} x_{i_2} \ldots x_{i_r}.$$

Definition 1.2 *Let $S = \{\vec{x} = (x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \ldots, n\}$. The Lagrangian of G, denoted by $\lambda(G)$, is defined as*

$$\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in S\}.$$

The value x_i is called the weight *of the vertex i. We call $\vec{x} = (x_1, x_2, \ldots, x_n) \in R^n$ a legal weighting for G if $\vec{x} \in S$. A vector $\vec{y} \in S$ is called an* optimal weighting *for G if $\lambda(G, \vec{y}) = \lambda(G)$.*

The following fact is easily implied by the definition of the Lagrangian.

Fact 1.1 *Let G_1, G_2 be r-graphs and $G_1 \subseteq G_2$. Then $\lambda(G_1) \leq \lambda(G_2)$.*

In [5], Motzkin and Straus provided the following simple expression for the Lagrangian of a 2-graph.

Theorem 1.2 *(Motzkin and Straus [5]) If G is a 2-graph in which a largest clique has order t then $\lambda(G) = \lambda(K_t^{(2)}) = \frac{1}{2}(1 - \frac{1}{t})$.*

An attempt to generalize the Motzkin-Straus theorem to hypergraphs is due to Sós and Straus [13]. Recently, in [9] and [10] Rota Buló and Pelillo generalized the Motzkin and Straus' result to r-graphs in some way using a continuous characterization of maximal cliques. Determining the Lagrangian of a general r-graph is non-trivial when $r \geq 3$. Indeed the obvious generalization of Motzkin and Straus' result is false because there are many examples of r-graphs that do not achieve their Lagrangian on any proper subhypergraph.

For distinct $A, B \in \mathbb{N}^{(r)}$ we say that A is less than B in the *colex ordering* if $max(A \triangle B) \in B$, where $A \triangle B = (A \backslash B) \cup (B \backslash A)$. For example we have $246 < 156$ in $\mathbb{N}^{(3)}$ since $max(\{2, 4, 6\} \triangle \{1, 5, 6\}) \in \{1, 5, 6\}$. In colex ordering, $123 < 124 < 134 < 234 < 125 < 135 < 235 < 145 < 245 < 345 < 126 < 136 < 236 < 146 < 246 < 346 < 156 < 256 < 356 < 456 < 127 < \cdots$. Note that the first $\binom{t}{r}$ r-tuples in the colex ordering of $\mathbb{N}^{(r)}$ are the edges of $[t]^{(r)}$. The following conjecture of Frankl and Füredi (if it is true) proposes a solution to the question mentioned at the beginning.

Conjecture 1.3 *(Frankl and Füredi [1]) The r-graph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest Lagrangian of all r-graphs with m edges. In particular, the r-graph with $\binom{t}{r}$ edges and the largest Lagrangian is $[t]^{(r)}$.*

This conjecture is true when $r = 2$ by Theorem 1.2. For the case $r = 3$, Talbot in [14] proved the following.

Theorem 1.4 *(Talbot [14]) Let m and t be integers satisfying $\binom{t-1}{3} \le m \le \binom{t-1}{3} + \binom{t-2}{2} - (t-1)$. Then Conjecture 1.3 is true for $r = 3$ and this value of m. Conjecture 1.3 is also true for $r = 3$ and $m = \binom{t}{3} - 1$ or $m = \binom{t}{3} - 2$.*

For the case $r = 3$, Tang, Peng, Zhang, and Zhao in [15] proved the following.

Theorem 1.5 *([15]) Let m and t be integers. Then Conjecture 1.3 is true for $r = 3$ and $m = \binom{t}{3} - 3$ or $m = \binom{t}{3} - 4$.*

In this paper, we extend Theorems 1.4 and 1.5 as follows.

Theorem 1.6 *Let m and t be positive integers satisfying $\binom{t-1}{3} \le m \le \binom{t-1}{3} + \binom{t-2}{2} - (t-3)$. Then Conjecture 1.3 is true for $r = 3$ and this value of m.*

Theorem 1.7 *Let m and t be positive integers satisfying $m = \binom{t}{3} - 5$. Then Conjecture 1.3 is true for $r = 3$ and this value of m.*

We also establish bounds for Lagrangians of some special 3-graphs, then using these results, we show that

Theorem 1.8 *Conjecture 1.3 holds when $r = 3$ and $m \le 71$.*

The truth of Frankl and Füredi's conjecture is not known in general for $r \ge 4$. In the case $r = 3$, combining Theorems 1.4, 1.5, 1.6, and 1.7, we know that Conjecture 1.3 holds if $\binom{t-1}{3} - 5 \le m \le \binom{t-1}{3} + \binom{t-2}{2} - (t-3)$. The case when $\binom{t-1}{3} + \binom{t-2}{2} - (t-4) \le m \le \binom{t}{3} - 6$ is still open in this conjecture. In [4], He, Peng, and Zhao verified Frankl and Füredi's conjecture for $m \le 50$ when $r = 3$ through the software Matlab, while Theorem 1.8 is proved based on mathematical bounds.

The proof of Theorem 1.6 will be given in Section 3 and the proofs of Theorems 1.7, 1.8, and related results will be given in Section 4. Next, we state some useful results.

2 Useful Results

We will impose one additional condition on any optimal weighting $\vec{x} = (x_1, x_2, \ldots, x_n)$ for an r-graph G:

$$\begin{aligned}&|\{i : x_i > 0\}| \text{ is minimal, i.e. if } \vec{y} \text{ is a legal weighting for } G \text{ satisfying} \\ &|\{i : y_i > 0\}| < |\{i : x_i > 0\}|, \text{ then } \lambda(G, \vec{y}) < \lambda(G).\end{aligned} \tag{1}$$

For an r-graph $G = (V, E)$ we denote the $(r-1)$-neighborhood of a vertex $i \in V$ by $E_i = \{A \in V^{(r-1)} : A \cup \{i\} \in E\}$. Similarly, we will denote the $(r-2)$-neighborhood of a pair of vertices $i, j \in V$ by $E_{ij} = \{B \in V^{(r-2)} : B \cup \{i, j\} \in E\}$. We denote the complement of E_i by $E_i^c = \{A \in V^{(r-1)} : A \cup \{i\} \in V^{(r)} \backslash E\}$. Also, we denote the complement of E_{ij} by $E_{ij}^c = \{B \in V^{(r-2)} : B \cup \{i, j\} \in V^{(r)} \backslash E\}$. Denote

$$E_{i \backslash j} = E_i \cap E_j^c.$$

When the theory of Lagrange multipliers is applied to find the optimum of $\lambda(G,\vec{x})$, subject to $\sum_{i=1}^{n} x_i = 1$, notice that $\lambda(E_i,\vec{x})$ corresponds to the partial derivative of $\lambda(G,\vec{x})$ with respect to x_i. The following lemma gives some necessary conditions of an optimal weighting for G.

Lemma 2.1 *(Frankl and Rödl [2]) Let $G=(V,E)$ be an r-graph on the vertex set $[n]$ and $\vec{x}=(x_1,x_2,\dots,x_n)$ be an optimal weighting for G with k $(\leq n)$ non-zero weights satisfying condition (1). Then for every $\{i,j\}\in[k]^{(2)}$, (a) $\lambda(E_i,\vec{x})=\lambda(E_j,\vec{x})=r\lambda(G)$, (b) there is an edge in E containing both i and j.*

Definition 2.1 *An r-graph $G=(V,E)$ on the vertex set $[n]$ is left-compressed if $j_1j_2\cdots j_r\in E$ implies $i_1i_2\cdots i_r\in E$ provided $i_p\leq j_p$ for every $p, 1\leq p\leq r$. Equivalently, an r-graph $G=(V,E)$ is left compressed if $E_{j\setminus i}=\emptyset$ for any $1\leq i<j\leq n$.*

Remark 2.2 *(a) In Lemma 2.1, part(a) implies that*

$$x_j\lambda(E_{ij},\vec{x})+\lambda(E_{i\setminus j},\vec{x})=x_i\lambda(E_{ij},\vec{x})+\lambda(E_{j\setminus i},\vec{x}).$$

In particular, if G is left-compressed, then

$$(x_i-x_j)\lambda(E_{ij},\vec{x})=\lambda(E_{i\setminus j},\vec{x})$$

for any i,j satisfying $1\leq i<j\leq k$ since $E_{j\setminus i}=\emptyset$.

(b) If G is left-compressed, then for any i,j satisfying $1\leq i<j\leq k$,

$$x_i-x_j=\frac{\lambda(E_{i\setminus j},\vec{x})}{\lambda(E_{ij},\vec{x})} \tag{2}$$

holds. If G is left-compressed and $E_{i\setminus j}=\emptyset$ for i,j satisfying $1\leq i<j\leq k$, then $x_i=x_j$.

(c) By (2), if G is left-compressed, then an optimal legal weighting $\vec{x}=(x_1,x_2,\dots,x_n)$ for G must satisfy

$$x_1\geq x_2\geq\ldots\geq x_n\geq 0. \tag{3}$$

Let $C_{r,m}$ denote the r-graph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$.

Lemma 2.3 *(Talbot [14]) For integers m,t, and r satisfying $\binom{t-1}{r}\leq m\leq\binom{t-1}{r}+\binom{t-2}{r-1}$, we have $\lambda(C_{r,m})=\lambda([t-1]^{(r)})$.*

Denote

$$\lambda_m^r=\max\{\lambda(G): G \text{ is an } r-\text{graph with } m \text{ edges}\}.$$

In [14], the following results are proved.

Lemma 2.4 *(Talbot [14]) Let m be a positive integer. Let G be a left-compressed 3-graph with m edges such that $\lambda(G)=\lambda_m^3$. Let $\vec{x}=(x_1,x_2,\dots,x_n)$ be an optimal weighting for G and k be the number of non-zero weights in $\vec{x}$, then*

$$|[k-1]^{(3)}\setminus E|\leq k-2.$$

Lemma 2.5 *(Talbot [14]) Let $G = (V, E)$ be a 3-graph with m edges such that $\lambda(G) = \lambda_m^3$. Let $\vec{x} = (x_1, x_2, \ldots, x_k)$ be an optimal weighting for G satisfying $x_1 \geq x_2 \geq \ldots \geq x_k > x_{k+1} = \ldots = x_n = 0$. Then*

$$|E| \geq \binom{k-1}{3} + \binom{k-2}{2} - (k-2).$$

We will use Lemmas 2.6 and 2.7 in our main results. Lemma 2.6 is a special case of Theorem 3.3 in [7].

Lemma 2.6 *(Peng, Tang, Zhao [7]) Let m and t be positive integers satisfying $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2}$. Let $G = (V, E)$ be a left-compressed 3-graph with m edges and t vertices satisfying $|[t-2]^{(2)} \setminus E_t| \geq |E_{(t-1)t}|$. Then $\lambda(G) \leq \lambda([t-1]^{(3)})$.*

The following lemma implies that we only need to consider left-compressed r-graphs when Conjecture 1.3 is explored.

Lemma 2.7 *(Talbot [14]) Let m, t be positive integers satisfying $m \leq \binom{t}{r} - 1$, then there exists a left-compressed r-graph G with m edges such that $\lambda(G) = \lambda_m^r$.*

The following result is also used in the proof of our main results.

Theorem 2.8 *(Peng and Zhao [8]) Let m and t be positive integers satisfying $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2}$. Let G be a 3-graph with m edges and contain a clique of order $t-1$. Then $\lambda(G) = \lambda([t-1]^{(3)})$.*

3 Proof of Theorems 1.6

Proof of Theorem 1.6. Let m and t be positive integers satisfying $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2} - (t-3)$. Let $G = (V, E)$ be a 3-graph with m edges such that $\lambda(G) = \lambda_m^3$. By Lemma 2.3, it is sufficient to show that $\lambda(G) \leq \lambda([t-1]^{(3)})$.

Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ be an optimal weighting for G and k be the number of positive weights in $\vec{x}$. We can assume that G is left-compressed by Lemma 2.7. So $x_1 \geq x_2 \geq \ldots \geq x_k > x_{k+1} = \ldots = x_n = 0$ by Remark 2.2(c). Since $\vec{x}$ has only k positive weights, we can assume that G is on vertex set $[k]$.

Now we proceed to show that $\lambda(G) \leq \lambda([t-1]^{(3)})$. If $\lambda(G) > \lambda([t-1]^{(3)})$, then $k \geq t$ since otherwise $k \leq t-1$ and then $\lambda(G) \leq \lambda([t-1]^{(3)})$. By Lemma 2.1(a), $k-1$ and k appear in some common edge $e \in E$. Recall that E is left-compressed, so $1(k-1)k \in E$. Define $b = max\{i : i(k-1)k \in E\}$. Because E is left-compressed, $E_{i \setminus j} = \emptyset$ for $1 \leq i < j \leq b$. Hence, by Remark 2.2(a), we have $x_1 = x_2 = \cdots = x_b$. Clearly, $b \leq k-4$. Since G is left-compressed and $1(k-1)k \in E$, then $|[k-2]^{(2)} \cap E_k| \geq 1$. If $k \geq t+1$, then applying Lemma 2.4, we have

$$\begin{aligned} m = |E| &= |E \cap [k-1]^{(3)}| + |[k-2]^{(2)} \cap E_k| + |E_{(k-1)k}| \\ &\geq \binom{t}{3} - (t-1) + 2 \\ &\geq \binom{t-1}{3} + \binom{t-2}{2} + 1, \end{aligned} \tag{4}$$

which contradicts to the assumption that $m \leq \binom{t-1}{3} + \binom{t-2}{2} - (t-3)$. Recall that $k \geq t$, so we have

$$k = t.$$

We need the following lemma whose proof follows the lines of Lemma 2.5 in [14].

Lemma 3.1 *Let G be a left-compressed 3-graph on the vertex set $[t]$ with m edges where $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2}$, and $\lambda(G) = \lambda_m^3$. Let $\vec{x} = (x_1, x_2, \ldots, x_t)$ be an optimal weighting for G. Then $|[t-1]^{(3)} \backslash E| \leq t-3$, or $\lambda(G) \leq \lambda([t-1]^{(3)})$.*

Assume Lemma 3.1 is proved, we continue the proof. If $|[t-1]^{(3)} \backslash E| \leq t-3$, we add all triples in $[t-1]^{(3)} \setminus E$ to E and let the new 3-graph be G_1. Then G_1 contains $[t-1]^{(3)}$, the number of edges in G_1 is at most $\binom{t-1}{3} + \binom{t-2}{2}$ and $\lambda(G_1) \geq \lambda(G)$. Applying Theorem 2.8, $\lambda(G_1) = \lambda([t-1]^{(3)})$. Therefore $\lambda(G) \leq \lambda([t-1]^{(3)})$. This proves Theorem 1.6. ■

We must now prove Lemma 3.1.

Proof of Lemma 3.1. We may assume that $x_t > 0$ since $x_t = 0$ implies that $\lambda(G_1) \leq \lambda([t-1]^{(3)})$. We define a new legal weighting $\vec{y}$ for G as follows. Let $y_i = x_i$ for $i \neq t-1, t$, $y_{t-1} = x_{t-1} + x_t$ and $y_t = 0$.

By Lemma 2.1(a), $\lambda(E_{t-1}, \vec{x}) = \lambda(E_t, \vec{x})$, so

$$\begin{aligned}
\lambda(G, \vec{y}) - \lambda(G, \vec{x}) &= x_t(\lambda(E_{t-1}, \vec{x}) - x_t\lambda(E_{t(t-1)}, \vec{x})) \\
&\quad -x_t(\lambda(E_t, \vec{x}) - x_{t-1}\lambda(E_{(t-1)t}, \vec{x})) - x_{t-1}x_t\lambda(E_{(t-1)t}, \vec{x}) \\
&= x_t(\lambda(E_{t-1}, \vec{x}) - \lambda(E_t, \vec{x})) - x_t^2 \sum_{i=1}^{b} x_i \\
&= -bx_1x_t^2. \qquad (5)
\end{aligned}$$

Since $y_t = 0$ we may remove all edges containing t from E to form a new 3-graph $\overline{G} = ([t], \overline{E})$ with $\lambda(\overline{G}, \vec{y}) = \lambda(G, \vec{y})$ and $|\overline{E}| = |E| - |E_t|$. We will show that if $|[t-1]^{(3)} \setminus E| \geq t-2$, then there exists a set of edges $F \subset [t-1]^{(3)} \setminus E$ satisfying

$$\lambda(F, \vec{y}) \geq bx_1x_t^2 \qquad (6)$$

Then, using (5), (6), the 3-graph $G' = ([t], E')$, where $E' = \overline{E} \cup F$, satisfies

$$\begin{aligned}
\lambda(G', \vec{y}) &= \lambda(\bar{G}, \vec{y}) + \lambda(F, \vec{y}) \\
&\geq \lambda(G, \vec{y}) + bx_1x_t^2 \\
&= \lambda(G, \vec{x}).
\end{aligned}$$

Since $\vec{y}$ has only $t-1$ positive weights, then $\lambda(G') \leq \lambda([t-1]^{(3)})$, and consequently $\lambda(G) \leq \lambda([t-1]^{(3)})$.

We must now construct the set of edges F satisfying (6). Applying Remark 2.2(a) by taking $i = 1, j = t-1$, we have

$$x_1 = x_{t-1} + \frac{\lambda(E_{1\backslash(t-1)}, \vec{x})}{\lambda(E_{1(t-1)}, \vec{x})}.$$

Let $C = [t-2]^{(2)} \setminus E_{t-1}$. Then $\lambda(E_{1\backslash(t-1)}, \vec{x}) = x_t \sum_{i=b+1}^{t-2} x_i + \lambda(C, \vec{x})$. Applying this and multiplying bx_t^2 to the above equation (note that $\lambda(E_{1(t-1)}, \vec{x}) = \sum_{i=2, i\neq t-1}^{t} x_i$), we have

$$bx_1x_t^2 = bx_{t-1}x_t^2 + \frac{bx_t^3 \sum_{i=b+1}^{t-2} x_i}{\sum_{i=2,i\neq t-1}^{t} x_i} + \frac{bx_t^2\lambda(C, \vec{x})}{\sum_{i=2,i\neq t-1}^{t} x_i}.$$

Since $x_1 \ge x_2 \ge \cdots \ge x_t$, then

$$bx_1 x_t^2 \le bx_{t-1} x_t^2 (1 + \frac{t-(b+2)}{t-3}) + \frac{bx_t \lambda(C, \vec{x})}{t-2}. \tag{7}$$

Define $\alpha = \lceil \frac{b|C|}{t-2} \rceil$ and $\beta = \lceil b(1 + \frac{t-(b+2)}{t-3}) \rceil$. Notice that $\lceil b(1 + \frac{t-(b+2)}{t-3}) \rceil \le t-2$ since $b \le t-2$, $f(b) = b(1 + \frac{t-(b+2)}{t-3})$ increases as b increases and $f(t-2) = t-2$. So $\beta \le t-2$. Let the set $F_1 \subset [t-1]^{(3)} \setminus E$ consist of the α heaviest edges in $[t-1]^{(3)} \setminus E$ containing the vertex $t-1$ (note that $|[t-2]^{(2)} \setminus E_{t-1}| = |C| \ge \alpha$). Recalling that $y_{t-1} = x_{t-1} + x_t$ we have

$$\lambda(F_1, \vec{y}) \ge \frac{bx_t \lambda(C, \vec{x})}{t-2} + \alpha x_{t-1} x_t^2.$$

So using (7)

$$\lambda(F_1, \vec{y}) - bx_1 x_t^2 \ge x_{t-1} x_t^2 (\alpha - \beta). \tag{8}$$

If $\alpha > \beta$, $\lambda(F_1, \vec{y}) - bx_{t-1} x_t^2 > 0$ so defining $F = F_1$ satisfies (6).

Assume $\alpha \le \beta$. Suppose that $|[t-1]^{(3)} \backslash E| \ge t-2$. So $|[t-1]^{(3)} \setminus E| \ge t-2 \ge \beta$ (recall that $\beta \le t-2$). Let F_2 consist of any $\beta - \alpha$ edges in $[t-1]^{(3)} \setminus (E \cup F_1)$ and define $F = F_1 \cup F_2$. Then since $\lambda(F_2, \vec{y}) \ge (\beta - \alpha) x_{t-1}^3$ and using (8)

$$\lambda(F, \vec{y}) - bx_{t-1} x_t^2 = \lambda(F_1, \vec{y}) - bx_{t-1} x_t^2 + \lambda(F_2, \vec{y}) \ge (\beta - \alpha) x_{t-1}^3 - x_{t-1} x_t^2 (\beta - \alpha) \ge 0.$$

This proves Lemma 3.1. ■

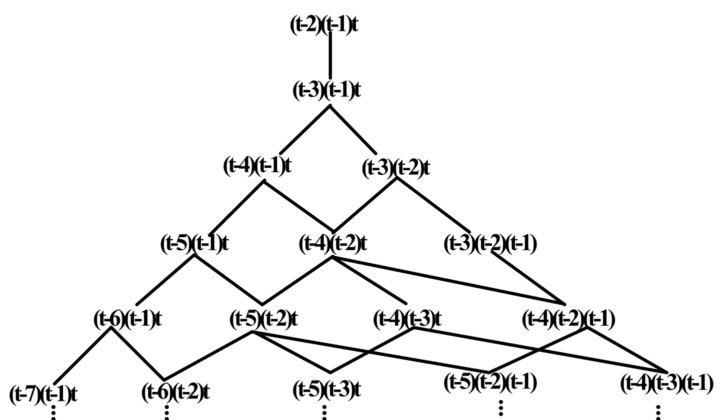

Figure 1

4 Proofs of Theorems 1.7 and 1.8

A triple $i_1 i_2 i_3$ is called a *descendant* of a triple $j_1 j_2 j_3$ if $i_s \le j_s$ for each $1 \le s \le 3$, and $i_1 + i_2 + i_3 < j_1 + j_2 + j_3$. In this case, the triple $j_1 j_2 j_3$ is called an *ancestor* of $i_1 i_2 i_3$. The triple $i_1 i_2 i_3$ is called a *direct descendant* of $j_1 j_2 j_3$ if $i_1 i_2 i_3$ is a descendant of $j_1 j_2 j_3$ and $j_1 + j_2 + j_3 = i_1 + i_2 + i_3 + 1$. We say that $j_1 j_2 j_3$ has lower hierarchy than $i_1 i_2 i_3$ if $j_1 j_2 j_3$ is an ancestor of $i_1 i_2 i_3$. This is a partial order on the set of all triples. Figure 1 is a Hessian diagram on all triples on vertex set $[t]$. In this diagram, $i_1 i_2 i_3$ and $j_1 j_2 j_3$ are connected by an edge if $i_1 i_2 i_3$ is a direct descendant of $j_1 j_2 j_3$.

Remark 4.1 *A 3-graph G is left-compressed if and only if all descendants of an edge of G are edges of G. Equivalently, if a 3-triple is not an edge of G, then none of its ancestors will be an edge of G.*

Lemma 4.2 *Let m and t be positive integers satisfying*

$$\binom{t-1}{3} \le m \le \binom{t-1}{3} + \binom{t-2}{2}.$$

Let $G = (V, E)$ be a left-compressed 3-graph on the vertex set $[t]$ with m edges satisfying $|E_{(t-1)t}| \le 3$. Then $\lambda(G) \le \lambda([t-1]^{(3)})$.

Proof. Because $\lambda(G)$ doesn't decrease as $|E|$ increases, we can assume that $|E| = \binom{t-1}{3} + \binom{t-2}{2}$. If E contains $[t-1]^{(3)}$ then $\lambda(G) = \lambda([t-1]^{(3)})$ by Theorem 2.8. Therefore, we can assume that G does not contain $[t-1]^{(3)}$ and $(t-3)(t-2)(t-1) \notin E$. If $|E_{(t-1)t}| = 1$, then $|[t-2]^{(2)} \backslash E_t| \ge |E_{(t-1)t}|$ since $(t-3)(t-2)t \notin E$. By Lemma 2.6, $\lambda(G) \le \lambda([t-1]^{(3)})$. If $t \le 5$ Lemma 4.2 clearly holds. Next, we assume $t \ge 6$ and distinguish two cases.

Case 1: $|E_{(t-1)t}| = 2$. Note that G is left-compressed, in view of Figure 1, $E = [t]^{(3)} \setminus \{3(t-1)t, 4(t-1)t, \cdots (t-2)(t-1)t, (t-3)(t-2)(t-1), (t-3)(t-2)t\}$. Let $E' = [t]^{(3)} \setminus \{3(t-1)t, 4(t-1)t, \cdots (t-2)(t-1)t, (t-4)(t-2)t, (t-3)(t-2)t\}$, and $G' = ([t], E')$. Assume $\vec{x} = (x_1, x_2, \ldots, x_t)$ is an optimal weighting for G satisfying $x_1 \ge x_2 \ge \ldots \ge x_t \ge 0$. By Remark 2.2(b), we have $x_1 = x_2$ and $x_{t-1} = x_t$. Note that $\vec{x}$ is also a legal weighting for G', and

$$\lambda(G', \vec{x}) - \lambda(G, \vec{x}) = x_{t-3}x_{t-2}x_{t-1} - x_{t-4}x_{t-2}x_t = (x_{t-3} - x_{t-4})x_{t-2}x_{t-1}. \tag{9}$$

Consider a new weighting for G': $\vec{y} = (y_1, y_2, \ldots, y_t)$ given by $y_i = x_i$ for $i \ne t-1$, $i \ne t$ and $y_{t-1} = x_{t-1} + \delta$, $y_t = x_t - \delta$. Lemma 2.1 implies that $\lambda(E_{t-1}, \vec{x}) = \lambda(E_t, \vec{x})$. So

$$\begin{aligned} \lambda(G', \vec{y}) - \lambda(G', \vec{x}) &= \delta(\lambda(E'_{t-1}, \vec{x}) - \lambda(E'_t, \vec{x})) - \delta^2 \lambda(E'_{(t-1)t}, \vec{x}) \\ &= \delta((\lambda(E_{t-1}, \vec{x}) + x_{t-3}x_{t-2}) - (\lambda(E_t, \vec{x}) - x_{t-4}x_{t-2})) - \delta^2 \lambda(E_{(t-1)t}, \vec{x}) \\ &= \delta(x_{t-4} + x_{t-3})x_{t-2} - 2\delta^2 x_1. \end{aligned} \tag{10}$$

Since $\lambda(E'_{(t-1)t}, \vec{x}) = \lambda(E_{(t-1)t}, \vec{x}) = x_1 + x_2 = 2x_1$. Let $\delta = \frac{(x_{t-4}+x_{t-3})x_{t-2}}{4x_1}$. By Remark 2.2(b), we have

$$\begin{aligned} x_{t-2} &= x_{t-1} + \frac{\lambda(E_{(t-2)\backslash(t-1)}, \vec{x})}{\lambda(E_{(t-2)(t-1)}, \vec{x})} \\ &= x_{t-1} + \frac{(x_3 + x_4 + \cdots + x_{t-4})x_t}{x_1 + x_2 + \cdots + x_{t-4}} \\ &\le x_{t-1} + \frac{t-6}{t-4}x_t \le 2x_t. \end{aligned} \tag{11}$$

So $\delta \le x_t$ and $\vec{y} = (y_1, y_2, \ldots, y_t)$ is also a legal weighting for G'. Replacing δ by $\frac{(x_{t-4}+x_{t-3})x_{t-2}}{4x_1}$ in (10), we have

$$\lambda(G', \vec{y}) - \lambda(G', \vec{x}) = \frac{(x_{t-4} + x_{t-3})^2 x_{t-2}^2}{8x_1}. \tag{12}$$

Again consider a new weighting for G': $\vec{z} = (z_1, z_2, \ldots, z_t)$ given by $z_i = x_i$ for $i \ne t-4, i \ne t-3$ and $z_{t-4} = y_{t-4} - \eta$, $z_{t-3} = y_{t-3} + \eta$. Then

$$\begin{aligned} \lambda(G', \vec{z}) - \lambda(G', \vec{y}) &= \eta(\lambda(E'_{t-3}, \vec{y}) - \lambda(E'_{t-4}, \vec{y})) - \eta^2 \lambda(E'_{(t-4)(t-3)}, \vec{y}) \\ &= \eta(y_{t-4} - y_{t-3})\lambda(E'_{(t-4)(t-3)}, \vec{y}) - \eta^2 \lambda(E'_{(t-4)(t-3)}, \vec{y}). \end{aligned} \tag{13}$$

Note that $y_{t-4} = x_{t-4}$, $y_{t-3} = x_{t-3}$, $\lambda(E'_{(t-4)(t-3)}, \vec{y}) = \lambda(E_{(t-4)(t-3)}, \vec{y}) = \lambda(E_{(t-4)(t-3)}, \vec{x})$. We have

$$\lambda(G', \vec{z}) - \lambda(G', \vec{y}) = \eta(x_{t-4} - x_{t-3})\lambda(E_{(t-4)(t-3)}, \vec{x}) - \eta^2\lambda(E_{(t-4)(t-3)}, \vec{x}). \tag{14}$$

Let $\eta = \frac{x_{t-4}-x_{t-3}}{2}$. Clearly, $z_{t-4} = z_{t-3} = \frac{x_{t-4}+x_{t-3}}{2}$, and $\vec{z}$ is also a legal weighting for G'. Replacing η by $\frac{x_{t-4}-x_{t-3}}{2}$ in (14), we have

$$\lambda(G', \vec{z}) - \lambda(G', \vec{y}) = \frac{(x_{t-4} - x_{t-3})^2}{4}\lambda(E_{(t-4)(t-3)}, \vec{x}). \tag{15}$$

Adding (9), (12) and (15), we have

$$\begin{aligned} \lambda(G', \vec{z}) - \lambda(G', \vec{x}) \quad = \quad & \frac{(x_{t-4} + x_{t-3})^2 x_{t-2}^2}{8x_1} + \frac{(x_{t-4} - x_{t-3})^2}{4}\lambda(E_{(t-4)(t-3)}, \vec{x}) \\ & -(x_{t-4} - x_{t-3})x_{t-2}x_{t-1}. \end{aligned} \tag{16}$$

By Remark 2.2(b), we have

$$\begin{aligned} x_{t-4} \quad &= \quad x_{t-3} + \frac{\lambda(E_{(t-4)\setminus(t-3)}, \vec{x})}{\lambda(E_{(t-4)(t-3)}, \vec{x})} \\ &= \quad x_{t-3} + \frac{(x_{t-1} + x_t)x_{t-2}}{\lambda(E_{(t-4)(t-3)}, \vec{x})}. \end{aligned} \tag{17}$$

Recall that $x_{t-1} = x_t$. Then

$$\frac{(x_{t-4} - x_{t-3})^2}{4}\lambda(E_{(t-4)(t-3)}, \vec{x}) = \frac{(x_{t-4} - x_{t-3})}{4}x_{t-2}(x_{t-1} + x_t) = \frac{(x_{t-4} - x_{t-3})}{2}x_{t-2}x_{t-1}. \tag{18}$$

And

$$\begin{aligned} x_{t-4} \quad &= \quad x_{t-3} + \frac{(x_{t-1} + x_t)x_{t-2}}{\lambda(E_{(t-4)(t-3)}, \vec{x})} \\ &= \quad x_{t-3} + \frac{(x_{t-1} + x_t)x_{t-2}}{x_1 + x_2 + \cdots + x_{t-5} + x_{t-2} + x_{t-1} + x_t} \\ &\leq \quad x_{t-3} + \frac{2}{t-2}x_{t-2}. \end{aligned} \tag{19}$$

So

$$(x_{t-4} - x_{t-3})x_{t-2}x_{t-1} \leq \frac{2}{t-2}x_{t-2}^2 x_{t-1}. \tag{20}$$

By Remark 2.2(b), we have

$$\begin{aligned} x_1 \quad &= \quad x_{t-3} + \frac{\lambda(E_{(1\setminus(t-3)}, \vec{x})}{\lambda(E_{1(t-3)}, \vec{x})} \\ &= \quad x_{t-3} + \frac{x_{t-1}x_t + x_{t-2}x_t + x_{t-2}x_{t-1}}{x_2 + x_3 + \cdots + x_{t-5} + x_{t-3} + x_{t-2} + x_{t-1} + x_t} \\ &\leq \quad x_{t-3} + \frac{3}{t-2}x_{t-2} \leq \frac{t+1}{t-2}x_{t-3}. \end{aligned} \tag{21}$$

Combing (16), (18), (20) and (21), recall that $t \geq 6$, we have

$$\lambda(G', \vec{z}) - \lambda(G, \vec{x}) \quad \geq \quad \frac{t-2}{2(t+1)}x_{t-3}x_{t-2}^2 - \frac{1}{t-2}x_{t-2}^2 x_{t-1} \geq 0. \tag{22}$$

So $\lambda(G,\vec{x}) \leq \lambda(G',\vec{z}) \leq \lambda(G') = \lambda([t-1]^{(3)})$, since G' contains a clique of order $t-1$ (by Theorem 2.8).

Case 2: $|E_{(t-1)t}| = 3$. In this case, since G is left-compressed, in view of Figure 1, we only need to consider $E = [t]^{(3)} \setminus \{4(t-1)t, \cdots (t-2)(t-1)t, (t-3)(t-2)(t-1), (t-3)(t-2)t, (t-4)(t-2)t\}$. Let $\bar{E} = [t]^{(3)} \setminus \{4(t-1)t, \cdots (t-2)(t-1)t, (t-4)(t-2)t, (t-4)(t-3)t, (t-3)(t-2)t\}$, and $\bar{G} = ([t], \bar{E})$. Assume that $\vec{\bar{x}} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_t)$ is an optimal weighting for G satisfying $\bar{x}_1 \geq \bar{x}_2 \geq \ldots \geq \bar{x}_t \geq 0$. By Remark 2.2(b), we have $\bar{x}_1 = \bar{x}_2 = \bar{x}_3$. Note that $\vec{\bar{x}}$ is also a legal weighting for $\bar{G}$, and

$$\lambda(\bar{G},\vec{\bar{x}}) - \lambda(G,\vec{\bar{x}}) = \bar{x}_{t-3}(\bar{x}_{t-2}\bar{x}_{t-1} - \bar{x}_{t-4}\bar{x}_t). \tag{23}$$

Consider a new weighting for $\bar{G}$: $\vec{\bar{y}} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_t)$ given by $\bar{y}_i = \bar{x}_i$ for $i \neq t-1$, $i \neq t$ and $\bar{y}_{t-1} = \bar{x}_{t-1} + \bar{\delta}$, $\bar{y}_t = \bar{x}_t - \bar{\delta}$. Then

$$\begin{aligned}
\lambda(\bar{G},\vec{\bar{y}}) - \lambda(\bar{G},\vec{\bar{x}}) &= \bar{\delta}(\lambda(\bar{E}_{t-1},\vec{\bar{x}}) - \lambda(\bar{E}_t,\vec{\bar{x}})) - \bar{\delta}^2\lambda(\bar{E}_{(t-1)t},\vec{\bar{x}}) \\
&= \bar{\delta}[(\lambda(E_{t-1},\vec{\bar{x}}) + \bar{x}_{t-3}\bar{x}_{t-2}) - (\lambda(E_t,\vec{\bar{x}}) - \bar{x}_{t-4}\bar{x}_{t-3})] - \bar{\delta}^2\lambda(\bar{E}_{(t-1)t},\vec{\bar{x}}) \\
&= \bar{\delta}(\bar{x}_{t-4} + \bar{x}_{t-2})\bar{x}_{t-3} - 3\bar{\delta}^2\bar{x}_1.
\end{aligned} \tag{24}$$

Let $\bar{\delta} = \frac{(\bar{x}_{t-4}+\bar{x}_{t-2})\bar{x}_{t-3}}{6\bar{x}_1}$. Using Remark 2.2(b), similar to Case 1, a tedious but easy calculation yields that $\bar{\delta} \leq \bar{x}_t$ and $\vec{\bar{y}} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_t)$ is also a legal weighting for G. Replacing $\bar{\delta}$ by $\frac{(\bar{x}_{t-4}+\bar{x}_{t-2})\bar{x}_{t-3}}{6\bar{x}_1}$ in (24), we have

$$\lambda(\bar{G},\vec{\bar{y}}) - \lambda(\bar{G},\vec{\bar{x}}) = \frac{(\bar{x}_{t-4}+\bar{x}_{t-2})^2\bar{x}_{t-3}^2}{12\bar{x}_1}. \tag{25}$$

Adding (23) and (25)

$$\lambda(\bar{G},\vec{\bar{y}}) - \lambda(G,\vec{\bar{x}}) = \frac{(\bar{x}_{t-4}+\bar{x}_{t-2})^2\bar{x}_{t-3}^2}{12\bar{x}_1} + \bar{x}_{t-3}(\bar{x}_{t-2}\bar{x}_{t-1} - \bar{x}_{t-4}\bar{x}_t). \tag{26}$$

By Remark 2.2(b), we have

$$\bar{x}_{t-1} = \bar{x}_t + \frac{\bar{x}_{t-4}\bar{x}_{t-2}}{3\bar{x}_1},$$

and

$$\bar{x}_{t-4} = \bar{x}_{t-2} + \frac{\bar{x}_{t-3}\bar{x}_t + \bar{x}_{t-3}\bar{x}_{t-1}}{\lambda(\bar{E}_{(t-4)(t-2)})} \leq \bar{x}_{t-2} + \frac{\bar{x}_{t-3}\bar{x}_t + \bar{x}_{t-3}\bar{x}_{t-1}}{3x_1}.$$

So

$$\begin{aligned}
\lambda(\bar{G},\vec{\bar{y}}) - \lambda(G,\vec{\bar{x}}) &= \frac{(\bar{x}_{t-4}+\bar{x}_{t-2})^2\bar{x}_{t-3}^2}{12\bar{x}_1} + \frac{\bar{x}_{t-4}\bar{x}_{t-3}\bar{x}_{t-2}^2}{3\bar{x}_1} - \bar{x}_{t-3}\bar{x}_t(\bar{x}_{t-4} - \bar{x}_{t-2}) \\
&\geq \frac{(\bar{x}_{t-4}+\bar{x}_{t-2})^2\bar{x}_{t-3}^2}{12\bar{x}_1} + \frac{\bar{x}_{t-4}\bar{x}_{t-3}\bar{x}_{t-2}^2}{3\bar{x}_1} - \frac{\bar{x}_{t-3}^2\bar{x}_t^2}{3\bar{x}_1} - \frac{\bar{x}_{t-3}^2\bar{x}_{t-1}\bar{x}_t}{3\bar{x}_1} \\
&\geq \frac{\bar{x}_{t-2}^2\bar{x}_{t-3}^2}{3\bar{x}_1} + \frac{\bar{x}_{t-4}\bar{x}_{t-3}\bar{x}_{t-2}^2}{3\bar{x}_1} - \frac{\bar{x}_{t-3}^2\bar{x}_t^2}{3\bar{x}_1} - \frac{\bar{x}_{t-3}^2\bar{x}_{t-1}\bar{x}_t}{3\bar{x}_1} \geq 0.
\end{aligned} \tag{27}$$

So $\lambda(G,\vec{\bar{x}}) \leq \lambda(\bar{G},\vec{\bar{y}}) \leq \lambda(\bar{G}) = \lambda([t-1]^{(3)})$, since $\bar{G}$ contains a clique of order $t-1$ (by Theorem 2.8). ■

Proof of Theorem 1.7. Let m and t be integers satisfying $m = \binom{t}{3} - 5$. Let $G = (V,E)$ be a 3-graph with m edges such that $\lambda(G) = \lambda_m^3$. Applying Lemma 2.7, we can assume that G is left-compressed. Let $\vec{x} = (x_1, x_2, \ldots, x_k)$ be an optimal weighting for G satisfying $x_1 \geq x_2 \geq \ldots \geq x_k > x_{k+1} = \ldots = x_n = 0$.

We claim that $k \leq t$. Otherwise $k \geq t+1$, and then Lemma 2.5 implies that

$$\begin{aligned} m = |E| &\geq \binom{k-1}{3} + \binom{k-2}{2} - (k-2) \\ &\geq \binom{t}{3} + \binom{t-1}{2} - (t-1) \\ &> \binom{t}{3} - 5 \end{aligned} \tag{28}$$

which contradicts to the assumption that $m = \binom{t}{3} - 5$. If $k < t$, then clearly $\lambda(G) \leq \lambda([t-1]^3)$. However, if we take a legal weighting $\vec{x} = (x_1, \ldots, x_t)$, where $x_1 = x_2 = \cdots = x_{t-2} = \frac{1}{t-1}$ and $x_{t-1} = x_t = \frac{1}{2(t-1)}$, then $\lambda(C_{3,m}, \vec{x}) > \lambda([t-1]^{(3)})$. This contradiction implies that $k \geq t$. Hence $k = t$. In view of Figure 1, there are only four different left-compressed 3-graphs with $m = \binom{t}{3} - 5$ edges on vertex set $[t]$. They are

$$G_1 = ([t], E^{(1)}) = [t]^{(3)} \setminus \{(t-2)(t-1)t, (t-3)(t-1)t, (t-4)(t-1)t, (t-5)(t-1)t, (t-6)(t-1)t\},$$

$$G_2 = ([t], E^{(2)}) = [t]^{(3)} \setminus \{(t-2)(t-1)t, (t-3)(t-1)t, (t-4)(t-1)t, (t-5)(t-1)t, (t-3)(t-2)t\},$$

$$G_3 = ([t], E^{(3)}) = [t]^{(3)} \setminus \{(t-2)(t-1)t, (t-3)(t-1)t, (t-4)(t-1)t, (t-3)(t-2)t, (t-4)(t-2)t\},$$

and

$$G_4 = ([t], E^{(4)}) = [t]^{(3)} \setminus \{(t-2)(t-1)t, (t-3)(t-1)t, (t-4)(t-1)t, (t-3)(t-2)t, (t-3)(t-2)(t-1)\}.$$

Clearly, G_1 is formed by taking the first m sets in the colex order of $\mathbb{N}^{(3)}$. So in order to prove Theorem 1.7, we only need to prove that the Lagrangian of G_1 is the largest one among these four 3-graphs.

First, we prove that $\lambda(G_1) \geq \lambda(G_2)$. Let $\vec{x} = (x_1, x_2, \ldots, x_t)$ be an optimal weighting for G_2 satisfying $x_1 \geq x_2 \geq \ldots \geq x_t > 0$. By Remark 2.2(b), we have

$$x_{t-2} = x_{t-1} + \frac{x_{t-4}x_t + x_{t-5}x_t}{\lambda(E^{(2)}_{(t-2)(t-1)}, \vec{x})},$$

and

$$x_{t-6} = x_{t-3} + \frac{x_{t-1}x_t + x_{t-2}x_t}{\lambda(E^{(2)}_{(t-6)(t-3)}, \vec{x})}.$$

So

$$\begin{aligned} \lambda(G_1, \vec{x}) - \lambda(G_2, \vec{x}) &= x_{t-3}x_{t-2}x_t - x_{t-6}x_{t-1}x_t \\ &= x_{t-3}[x_{t-1} + \frac{x_{t-4}x_t + x_{t-5}x_t}{\lambda(E^{(2)}_{(t-2)(t-1)}, \vec{x})}]x_t - x_{t-6}x_{t-1}x_t \\ &= \frac{x_{t-4}x_t + x_{t-5}x_t}{\lambda(E^{(2)}_{(t-2)(t-1)}, \vec{x})}x_{t-3}x_t - (x_{t-6} - x_{t-3})x_{t-1}x_t \\ &= \frac{x_{t-4}x_t + x_{t-5}x_t}{\lambda(E^{(2)}_{(t-2)(t-1)}, \vec{x})}x_{t-3}x_t - \frac{x_{t-1}x_t + x_{t-2}x_t}{\lambda(E^{(2)}_{(t-6)(t-3)}, \vec{x})}x_{t-1}x_t \\ &\geq 0 \end{aligned} \tag{29}$$

since $\lambda(E^{(2)}_{(t-2)(t-1)}, \vec{x}) \leq \lambda(E^{(2)}_{(t-6)(t-3)}, \vec{x})$. Hence $\lambda(G_1) \geq \lambda(G_1, \vec{\vec{x}}) \geq \lambda(G_2, \vec{x}) \geq \lambda(G_2)$.

Next, we prove $\lambda(G_2) \geq \lambda(G_3)$. Let $\vec{y} = (y_1, y_2, \ldots, y_t)$ be an optimal weighting for G_3 satisfying $y_1 \geq y_2 \geq \ldots \geq y_t \geq 0$. Note that $y_{t-2} = y_{t-1}$, $\vec{y}$ is also a legal weighting for G_2, and

$$\lambda(G_2, \vec{y}) - \lambda(G_3, \vec{y}) = (y_{t-4} - y_{t-5}) y_{t-1} y_t. \tag{30}$$

Consider a new weighting for G_2, $\vec{y'} = (y'_1, y'_2, \ldots, y'_t)$ given by $y'_i = y_i$ for $i \neq t-5$, $i \neq t-4$ and $y'_{t-5} = y_{t-5} - \delta$, $y'_{t-4} = y_{t-4} + \delta$. Then

$$\begin{aligned} \lambda(G_2, \vec{y'}) - \lambda(G_2, \vec{y}) &= \delta(\lambda(E^{(2)}_{t-4}, \vec{y}) - \lambda(E^{(2)}_{t-5}, \vec{y})) - \delta^2 \lambda(E^{(2)}_{(t-5)(t-4)}, \vec{y}) \\ &= \delta(y_{t-5} - y_{t-4}) \lambda(E^{(2)}_{(t-5)(t-4)}, \vec{y}) - \delta^2 \lambda(E^{(2)}_{(t-5)(t-4)}, \vec{y}). \end{aligned}$$

Let $\delta = \frac{y_{t-5} - y_{t-4}}{2}$. Clearly, $\vec{y'} = (y'_1, y'_2, \ldots, y'_t)$ is also a legal weighting for G_2. And

$$\lambda(G_2, \vec{y'}) - \lambda(G_2, \vec{y}) = \frac{(y_{t-5} - y_{t-4})^2}{4} \lambda(E^{(2)}_{(t-5)(t-4)}, \vec{y}). \tag{31}$$

Let $\vec{y''} = (y''_1, y''_2, \ldots, y''_t)$ given by $y''_i = y'_i$ for $i \neq t-2$, $i \neq t-1$ and $y''_{t-2} = y'_{t-2} + \eta$, $y''_{t-1} = y'_{t-1} - \eta$. Then

$$\begin{aligned} \lambda(G_2, \vec{y''}) - \lambda(G_2, \vec{y'}) &= \eta[\lambda(E^{(2)}_{t-2}, \vec{y'}) - \lambda(E^{(2)}_{t-1}, \vec{y'})] - \eta^2 \lambda(E^{(2)}_{(t-2)(t-1)}, \vec{y'}) \\ &= \eta(y'_{t-4} y'_t + y'_{t-5} y'_t) - \eta^2 \lambda(E^{(2)}_{(t-2)(t-1)}, \vec{y'}). \end{aligned} \tag{32}$$

Let $\eta = \frac{y'_{t-4} y'_t + y'_{t-5} y'_t}{2\lambda(E^{(2)}_{(t-2)(t-1)}, \vec{y'})}$. Clearly, $\eta < y'_t$. Hence, $\vec{y''} = (y''_1, y''_2, \ldots, y''_t)$ is also a legal weighting for G_2. And

$$\lambda(G_2, \vec{y''}) - \lambda(G_2, \vec{y'}) = \frac{(y'_{t-5} + y'_{t-4})^2 y'^2_t}{4\lambda(E^{(2)}_{(t-2)(t-1)}, \vec{y})}. \tag{33}$$

By Remark 2.2(b), we have

$$y_{t-5} = y_{t-4} + \frac{2 y_{t-1} y_t}{\lambda(E^{(3)}_{(t-5)(t-4)}, \vec{y})}. \tag{34}$$

Combing (30), (31), (33), and (34), we have

$$\begin{aligned} \lambda(G_2, \vec{y''}) - \lambda(G_3, \vec{y}) &= \frac{y^2_{t-1} y^2_t}{\lambda(E^{(3)}_{(t-5)(t-4)}, \vec{y})} \\ &+ \frac{(y'_{t-5} + y'_{t-4})^2 y'^2_t}{4\lambda(E^{(2)}_{(t-2)(t-1)}, \vec{y'})} - \frac{2 y^2_{t-1} y^2_t}{\lambda(E^{(3)}_{(t-5)(t-4)}, \vec{y})}. \end{aligned}$$

Since $\lambda(E^{(2)}_{(t-2)(t-1)}, \vec{y'}) = \lambda(E^{(3)}_{(t-2)(t-1)}, \vec{y})$, $y'_{t-5} + y'_{t-4} = y_{t-5} + y_{t-4}$, $y'_t = y_t$ and $\lambda(E^{(3)}_{(t-2)(t-1)}, \vec{y}) \leq \lambda(E^{(3)}_{(t-5)(t-4)}, \vec{y})$, we have $\lambda(G_2, \vec{y''}) - \lambda(G_3, \vec{y}) \geq 0$. Hence $\lambda(G_2) \geq \lambda(G_2, \vec{y''}) \geq \lambda(G_3, \vec{y}) = \lambda(G_3)$.

Last, we prove $\lambda(G_3) \geq \lambda(G_4)$. Let $\vec{z} = (z_1, z_2, \ldots, z_t)$ be an optimal weighting for G_4 satisfying $z_1 \geq z_2 \geq \ldots \geq z_t \geq 0$. By Remark 2.2(b), $z_{t-1} = z_t$, and

$$\lambda(G_3, \vec{z}) - \lambda(G_4, \vec{z}) = (z_{t-3} - z_{t-4}) z_{t-2} z_{t-1}. \tag{35}$$

Consider a new weighting for G_3, $\vec{z'} = (z'_1, z'_2, \ldots, z'_t)$ given by $z'_i = z_i$ for $i \neq t-4$, $i \neq t-3$ and $z'_{t-4} = z_{t-4} - \alpha$, $z'_{t-3} = z_{t-3} + \alpha$. Then

$$\begin{aligned}\lambda(G_3, \vec{z'}) - \lambda(G_3, \vec{z}) &= \alpha(\lambda(E^{(3)}_{t-3}, \vec{z}) - \lambda(E^{(3)}_{t-4}, \vec{z})) - \alpha^2\lambda(E^{(3)}_{(t-4)(t-3)}, \vec{y}) \\ &= \alpha(z_{t-4} - z_{t-3})\lambda(E^{(3)}_{(t-4)(t-3)}, \vec{z}) - \alpha^2\lambda(E^{(3)}_{(t-4)(t-3)}, \vec{z}). \qquad (36)\end{aligned}$$

Let $\alpha = \frac{z_{t-4} - z_{t-3}}{2}$. Clearly, $\vec{z'} = (z'_1, z'_2, \ldots, z'_t)$ is also a legal weighting for G_3. Also

$$\lambda(G_3, \vec{z'}) - \lambda(G_3, \vec{z}) = \frac{(z_{t-4} - z_{t-3})^2}{4}\lambda(E^{(3)}_{(t-4)(t-3)}, \vec{z}). \qquad (37)$$

Let $\vec{z''} = (z''_1, z''_2, \ldots, z''_t)$ be given by $z''_i = z'_i$ for $i \neq t-1$, $i \neq t$ and $z''_{t-1} = z'_{t-1} + \beta$, $z''_t = z'_t - \beta$. Then

$$\begin{aligned}\lambda(G_3, \vec{z''}) - \lambda(G_3, \vec{z'}) &= \beta(\lambda(E^{(3)}_{t-1}, \vec{z'}) - \lambda(E^{(3)}_{t}, \vec{z'})) - \beta^2\lambda(E^{(3)}_{(t-1)t}, \vec{z'}) \\ &= \beta(z'_{t-3}z'_{t-2} + z'_{t-4}z'_{t-2}) - \beta^2\lambda(E^{(3)}_{(t-1)t}, \vec{z'}). \qquad (38)\end{aligned}$$

Let $\beta = \frac{z'_{t-3}z'_{t-2} + z'_{t-4}z'_{t-2}}{2\lambda(E^{(3)}_{(t-1)t}, \vec{z'})}$. Since $1, 2 \in E_{(t-1)t}$, $\beta \leq \frac{z'_{t-2}}{2} \leq z'_t$. Hence, $\vec{z''} = (z''_1, z''_2, \ldots, z''_t)$ is also a legal weighting for G_3, and

$$\lambda(G_3, \vec{z''}) - \lambda(G_3, \vec{z'}) = \frac{(z'_{t-4} + z'_{t-3})^2 z'^2_{t-2}}{4\lambda(E^{(3)}_{(t-1)t}, \vec{z'})}. \qquad (39)$$

By Remark 2.2(b), we have

$$z_{t-4} = z_{t-3} + \frac{2z_{t-2}z_t}{\lambda(E^{(4)}_{(t-4)(t-3)}, \vec{z})}. \qquad (40)$$

Note that $\lambda(E^{(4)}_{(t-4)(t-3)}, \vec{z}) = \lambda(E^{(3)}_{(t-4)(t-3)}, \vec{z})$, $z'_{t-4} + z'_{t-3} = z_{t-4} + z_{t-3}$, $z'_{t-2} = z_{t-2}$. Combing (35), (37),(39) and (40), we have

$$\begin{aligned}\lambda(G_3, \vec{z''}) - \lambda(G_4, \vec{z}) &= \frac{(z_{t-4} - z_{t-3})^2\lambda(E^{(3)}_{(t-4)(t-3)}, \vec{z})}{4} + \frac{(z'_{t-4} + z'_{t-3})^2 z'^2_{t-2}}{4\lambda(E^{(3)}_{(t-1)t}, \vec{z'})} \\ &\quad - \frac{2z^2_{t-2}z^2_t}{\lambda(E^{(3)}_{(t-4)(t-3)}, \vec{z})} \\ &= \frac{z^2_{t-2}z^2_t}{\lambda(E^{(3)}_{(t-4)(t-3)}, \vec{z})} + \frac{(z_{t-4} + z_{t-3})^2 z^2_{t-2}}{4\lambda(E^{(3)}_{(t-1)t}, \vec{z'})} \\ &\quad - \frac{2z^2_{t-2}z^2_t}{\lambda(E^{(3)}_{(t-4)(t-3)}, \vec{z})} \geq 0 \qquad (41)\end{aligned}$$

since $\lambda(E^{(3)}_{(t-4)(t-3)}, \vec{z}) = \lambda(E^{(3)}_{(t-4)(t-3)}, \vec{z'}) \geq \lambda(E^{(3)}_{(t-1)t}, \vec{z'})$. Hence $\lambda(G_3) \geq \lambda(G_3, \vec{z''}) \geq \lambda(G_4, \vec{z}) = \lambda(G_4)$. This completes the proof of Theorem 1.7. ∎

Proof of Theorem 1.8. We prove this theorem by case analysis.

Note that $m = 1 = \binom{3}{3}, m = 2 = \binom{4}{3} - 2, m = 3 = \binom{4}{3} - 1, m = 4 = \binom{4}{3}, m = 8 = \binom{5}{3} - 2, m = 9 = \binom{5}{3} - 1, m = 18 = \binom{6}{3} - 2, m = 19 = \binom{6}{3} - 1, m = 33 = \binom{7}{3} - 2, m = 34 = \binom{7}{3} - 1$, and

$m = 54 = \binom{8}{3} - 2, m = 55 = \binom{8}{3} - 1, m = 56 = \binom{8}{3}$. Conjecture 1.3 holds for 3-graph with the number of edges equal these values by Lemma 2.7. Since $m = 6 = \binom{5}{3} - 4, m = 7 = \binom{5}{3} - 3, m = 16 = \binom{6}{3} - 4, m = 17 = \binom{6}{3} - 3, m = 31 = \binom{7}{3} - 4, m = 32 = \binom{7}{3} - 3, m = 52 = \binom{8}{3} - 4, m = 53 = \binom{8}{3} - 3$, Conjecture 1.3 holds for 3-graph with the number of edges equal these values by Theorem 1.5. Because $m = 5 = \binom{5}{3} - 5, m = 51 = \binom{8}{3} - 5$, Conjecture 1.3 holds for 3-graph with the number of edges equal these values by Theorem 1.7.

If $\binom{6-1}{3} \leq m = 10, 11, 12, 13, 14, 15 \leq \binom{6-1}{3} + \binom{6-2}{2}$, $\lambda(C_{3,m}) = \lambda([5]^{(3)})$. Let $G = (V, E)$ be a 3-graph satisfying $\lambda(G) = \lambda^3_m$. By Lemma 2.7, we can assume that G is left-compressed. Let $\vec{x} = (x_1, x_2, \ldots, x_t)$ be an optimal weighting for G, by Lemma 1.4, $x_i = 0$ for $i \geq 7$. Otherwise $|E| \geq \binom{6}{3} + \binom{5}{2} - 5 = 25 > m$. So we can assume G is on vertex set $[6] = \{1, 2, 3, 4, 5, 6\}$. We claim edge 456 is not in E. Otherwise, if 456 is in E, then all its descendants are in E, $E \supseteq [6]^{(3)}$. So $|E| \geq 20$, this contradicts to $|E| \leq 15$. Hence $|E_{56}| < 4$ and we have $\lambda(G) \leq \lambda([5]^{(3)})$ by Lemma 4.2.

If $\binom{7-1}{3} \leq m = 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30 \leq \binom{7-1}{3} + \binom{7-2}{2}$, $\lambda(C_{3,m}) = \lambda([6]^{(3)})$. We can assume G is on vertex set $[7] = \{1, 2, 3, 4, 5, 6, 7\}$. We claim 467 is not in E. Otherwise, if 467 is in E, then all its descendants are in E, $E \supseteq [7]^{(3)} \setminus \{567\}$. So $|E| \geq 35 - 1 = 34$, this contradicts to $|E| \leq 30$. Hence $|E_{67}| < 4$ and we have $\lambda(G) \leq \lambda([6]^{(3)})$ by Lemma 4.2.

If $\binom{8-1}{3} \leq m = 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 47, 48, 49, 50 \leq \binom{8-1}{3} + \binom{8-2}{2}$, $\lambda(C_{3,m}) = \lambda([7]^{(3)})$. We can assume G is on vertex set $[8] = \{1, 2, 3, 4, 5, 6, 7, 8\}$. We claim 478 is not in E. Otherwise, if 478 is in E, then all its descendants are in E, $E \supseteq [8]^{(3)} \setminus \{678, 578, 568, 567\}$. So $|E| \geq 56 - 4 = 52$, contradicts to $|E| \leq 50$. Hence $|E_{78}| < 4$ and we have $\lambda(G) \leq \lambda([7]^{(3)})$ by Lemma 4.2.

If $56 = \binom{9-1}{3} \leq m \leq \binom{9-1}{3} + \binom{9-2}{2} - (9-3) = 71$, Theorem 1.6 implies Theorem 1.8.

This completes the proof. ■

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